J. Fluid Mech. (1966), *vol.* 24, *part* 2, *pp.* 285–291
Printed in Great Britain

Boundary effects in Stokes flow

By W. E. WILLIAMS
Department of Applied Mathematics, The University of Liverpool†

(Received 22 March 1965)

An integral-equation approach is used to calculate the first-order effect of the proximity of a boundary on the Stokes resistance of a body. It is shown that this approach gives both a simple derivation of the modified Stokes resistance and a method of investigating the accuracy of the final result. The problem of two settling particles is also considered briefly.

1. Introduction

In two recent papers Brenner (1962, 1964) has obtained general formulae for the effect of the proximity of a boundary on the Stokes resistance of an arbitrary body. Brenner's formulae are valid for small values of a parameter c/l (c = characteristic body dimension, l = minimum distance between a point on the body and a point on the boundary) and were derived by the 'method of reflexions' used by Brenner & Happel (1958) in earlier work on similar problems.

In the first of the two papers cited above, the problem considered is that of a body moving parallel to one of its principal axes of resistance, and it is deduced that the force F on the body is given by

$$F/F_\infty = (1 - kF_\infty/6\pi\mu Ul)^{-1},$$

where F_∞ is the corresponding force in an unbounded medium, μ is the viscosity, U is the body velocity and k is a constant independent of the particular form of the body. The advantage of the above result is that if k can be determined by direct calculation of F for one particular body then this value of k may then be used to give F for any other body, provided that the force in an unbounded fluid is known. Brenner's derivation indicates that the error in the above formula is $O(c^2/l^2)$ but he suggests that it is in fact $O(c^3/l^3)$. He offers no proof of this, though all known solutions exhibit this type of behaviour.

Similar formulae to the above have recently been derived by the author (Williams 1964) for the effects of a boundary on the capacity of a conductor. In this type of problem it was found that an integral-equation approach gave a simple method of calculating the capacity and it was possible to estimate the error very accurately. It would therefore be of interest to investigate whether an integral-equation method would be useful for determining the accuracy of the above (and related) formulae. The general integral formulation of Stokes

† Present address: Department of Mathematics, Battersea College of Technology, London, S.W. 11.

flow problems is rather more complicated than that of electrostatic problems but can be carried out completely by using formulae derived by Lorentz (cf. Oseen 1927). The object of the present paper is to present this approach to the calculation of boundary effects and to the estimation of the errors made in the final formulae.

The general problem reduces to that of solving a set of integral equations, and it is shown that the approximate solution of these equations can be carried out completely provided the solution for the Stokes flow problem in an unbounded medium is known. The necessary calculations are fairly simple and the results obtained by Brenner in his two papers are derived in an elementary fashion. Furthermore, it is shown that, if certain symmetry restrictions are imposed, Brenner's formulae are correct to $O(c^3/l^3)$. The problem of two arbitrary particles settling in a force field is very briefly examined and the first-order effect of particle proximity on settling velocity is obtained.

2. Derivation of integral representations

In the Stokes flow of an incompressible fluid the fluid velocity $\mathbf{q}$ and the pressure p satisfy the equations

$$\mu\nabla^2\mathbf{q} = \operatorname{grad} p, \quad \operatorname{div}\mathbf{q} = 0, \tag{1}$$

where μ is the viscosity. Integral expressions for the solution of equations (1) which are analogous to Green's formulae in potential theory were first derived by Lorentz (cf. Oseen 1927), and in order to present his formulae it is necessary to define certain tensor functions which are the appropriate generalizations of the Green's function of potential theory.

The tensor $\mathbf{T}_1(\mathbf{r}, \mathbf{r}_0)$, where $\mathbf{r}$ and $\mathbf{r}_0$ are the position vectors of two arbitrary points P and P_0 respectively, is defined by

$$\mathbf{T}_1 = \mathbf{U}\nabla^2|\mathbf{r}-\mathbf{r}_0| - \operatorname{grad}\operatorname{grad}|\mathbf{r}-\mathbf{r}_0|, \tag{2}$$

where $\mathbf{U}$ is the unit tensor, and the vector $\mathbf{p}_1$ is defined by

$$\mathbf{p}_1 = -\mu\operatorname{grad}\nabla^2|\mathbf{r}-\mathbf{r}_0|.$$

It is easily established that

$$\mu\nabla^2\mathbf{T}_1 - \operatorname{grad}\mathbf{p}_1 = -8\pi\mu\mathbf{U}\delta(\mathbf{r}-\mathbf{r}_0), \quad \operatorname{div}\mathbf{T}_1 = 0, \tag{3}$$

and it follows from equation (3) that $\mathbf{i}.\mathbf{T}_1$, where $\mathbf{i}$ is an arbitrary unit vector, is the velocity at P due to a point force acting parallel to $\mathbf{i}$ at P_0. (Note that $\mathbf{i}.\mathbf{p}_1$ is the pressure p at P.) In the region enclosed by any given closed surface S_2 we now define a second tensor $\mathbf{T}$ and a corresponding vector $\mathbf{p}$ so that they satisfy equation (3), with the additional condition that $\mathbf{T} = 0$ on S_2. If both P and P_0 are within S_2 then $\mathbf{i}.\mathbf{T}$ may be interpreted as the velocity at P in the region enclosed by the rigid surface S_2 when a point force parallel to $\mathbf{i}$ is acting at P_0. Clearly $\mathbf{T}$ and $\mathbf{p}$ may be written as $\mathbf{T}_1+\mathbf{T}_2$ and $\mathbf{p}_1+\mathbf{p}_2$, respectively, where $\mathbf{T}_2$, $\mathbf{p}_2$ satisfy equations (3) with the right-hand side of the first equation set equal to zero. It will be assumed for the present that $\mathbf{T}_2$ may be determined explicitly for any particular surface S_2. The boundary-value problem is a well-defined one, but the actual determination of $\mathbf{T}_2$ for any given surface S_2 is rather complicated and the only case for which $\mathbf{T}_2$ has been completely determined is when S_2 is an infinite plane (Oseen 1927).

It is now possible to write down, from Lorentz's work, an expression for $\mathbf{q}(P)$, the solution of equations (1) at any point P in the region between two closed surfaces S_1 and S_2 with S_1 completely enclosed in S_2, such that $\mathbf{q}$ vanishes on S_2. The appropriate result is

$$\mathbf{q}(P) = -\frac{1}{8\pi\mu}\int_{S_1}\left\{\left(\mu\frac{\partial \mathbf{q}}{\partial n} - p\mathbf{n}\right).\mathbf{T} - \mathbf{q}.\left(\mu\frac{\partial \mathbf{T}}{\partial n} - \mathbf{np}\right)\right\}dS, \tag{4}$$

where P_0 is now a variable point on S_1 and $\mathbf{n}$ denotes the outward normal to S_1. For the case when S_2 recedes to infinity $\mathbf{T}$ and $\mathbf{p}$ in equation (4) will be replaced by $\mathbf{T}_1$ and $\mathbf{p}_1$, respectively. Furthermore, it follows from Green's theorems and equations (3) that, if $\mathbf{q}$ is constant on S_1, equation (4) reduces to

$$\mathbf{q}(P) = -\frac{1}{8\pi\mu}\int_{S_1}\mathbf{f}.\mathbf{T}\,dS, \tag{5}$$

where $\mathbf{f} = \mu(\partial\mathbf{q}/\partial n) - p\mathbf{n}$.

The above formulae may be applied immediately to the problem of the fluid motion generated by the rigid body S_1 moving with uniform velocity $\mathbf{V}$ in viscous fluid in the region bounded by the rigid surface S_2. Clearly the fluid velocity will be given by equation (5) and it also follows from the definition of the stress tensor that the force $\mathbf{F}$ on the body is given by

$$\mathbf{F} = \int_{S_1}\mathbf{f}dS. \tag{6}$$

The force $\mathbf{F}_\infty$ which S_1 would experience in an unbounded medium may be expressed in terms of the Stokes resistance tensor $\boldsymbol{\phi}_\infty$ defined by Brenner (1963) as $\mathbf{F}_\infty = -6\pi\mu c\mathbf{V}.\boldsymbol{\phi}_\infty$, where c is a characteristic dimension of S_1. It is also convenient at this stage to define a parameter ϵ as the ratio of c to the minimum distance between a point of S_1 and a point of S_2. If the origin O is taken to be some point of S_1 and if P and P_0 are also both on S_1, then it follows that, neglecting terms of $O(\epsilon^3)$,

$$\mathbf{T}_2 = \mathbf{T}_2^0 + \mathbf{r}.[\operatorname{grad}\mathbf{T}_2]_{\mathbf{r}=\mathbf{r}_0=0} + \mathbf{r}_0.[\operatorname{grad}^0\mathbf{T}_2]_{\mathbf{r}=\mathbf{r}_0=0}, \tag{7}$$

where $\mathbf{T}_2^0 = T_2(0,0)$ and the affix zero implies differentiation with respect to the components of $\mathbf{r}_0$.

The boundary-value problem for flow past S_1 thus reduces to the solution of the set of integral equations

$$\mathbf{V} = -\frac{1}{8\pi\mu}\int_{S_1}\mathbf{f}.(\mathbf{T}_1+\mathbf{T}_2)\,dS, \quad P \text{ and } P_0 \text{ both on } S_1. \tag{8}$$

It follows that, if $\mathbf{T}_2$ in equation (8) is replaced by $\mathbf{T}_2^0$ [i.e. neglecting the contributions of the second and third terms on the right-hand side of equation (7) which are of $O(\epsilon^2)$], then

$$\mathbf{V} + \frac{1}{8\pi\mu}\mathbf{F}.\mathbf{T}_2^0 = -\frac{1}{8\pi\mu}\int_{S_1}\mathbf{f}.\mathbf{T}_1\,dS. \tag{9}$$

Equation (9) is the integral equation for Stokes flow when S_1 is moving in unbounded fluid with the constant velocity $\mathbf{V}+(1/8\pi\mu)\mathbf{F}.\mathbf{T}_2^0$. It follows from equation (9) that

$$\mathbf{F} = -6\pi\mu c\left(\mathbf{V}+\frac{1}{8\pi\mu}\mathbf{F}.\mathbf{T}_2^0\right).\boldsymbol{\phi}_\infty, \tag{10}$$

and the solution of equation (10) is

$$\mathbf{F} = -6\pi\mu c\mathbf{V}.[\boldsymbol{\phi}_\infty^{-1}+\tfrac{3}{4}c\mathbf{T}_2^0]^{-1}. \tag{11}$$

Equation (11) is the basic result derived by Brenner (1964) using a more elaborate approach. It follows immediately from equation (10) that, if $\mathbf{F}$ on the right-hand side is replaced by $\mathbf{F}_\infty$,

$$\mathbf{F} = -6\pi\mu c\left(\mathbf{V}+\frac{1}{8\pi\mu}\mathbf{F}_\infty.\mathbf{T}_2^0\right).\boldsymbol{\phi}_\infty, \tag{12}$$

the error in equation (12) being $O(\epsilon^2)$.

The above equations may be simplified further if $\mathbf{V}$ is parallel to one of the principal axes of resistance of S_1; these axes being defined so that, in motion through an unbounded fluid parallel to one of them, the force is in the direction of motion. In this case equation (10) now becomes

$$F/F_\infty = 1/(1-\lambda F_\infty), \tag{13}$$

where λ is independent of the form of S_2, and equation (13) may be identified with a result obtained by Brenner (1962). For the case when S_1 is moving parallel to a principal axis of resistance it follows that equation (12) becomes

$$F/F_\infty = 1+\lambda F_\infty. \tag{14}$$

λF_∞ is of $O(\epsilon)$ and hence, neglecting terms of $O(\epsilon^2)$, equations (13) and (14) are identical, as are equations (11) and (12). Thus in general there is no merit in using equations (11) or (13) rather than the simpler equations (12) and (14). If, however, equation (10) is correct [neglecting terms of $O(\epsilon^3)$] then equations (11) and (13), which are exact solutions of equation (10), will also be accurate to this order. In fact in his derivation of equation (13) Brenner (1962) states that 'arguments too lengthy to give here suggest that the error does not exceed $O(\epsilon^3)$'.

We shall now investigate some of the simpler situations in which the error term in equation (10) is $O(\epsilon^3)$. This will clearly be the situation if

$$\int_{S_1} \mathbf{f}^\infty.[\mathbf{r}.[\mathrm{grad}\,\mathbf{T}_2]_{\mathbf{r}=\mathbf{r}_0=0}+\mathbf{r}_0.[\mathrm{grad}^0\,\mathbf{T}_2]_{\mathbf{r}=\mathbf{r}_0=0}]\,dS_1 = 0, \tag{15}$$

where $\mathbf{f}^\infty$ is the solution for $\mathbf{f}$ for the Stokes flow past S_1 in unbounded fluid. The simplest situation for which equation (15) holds is when S_2 is such that there exists a point O on S_1 through which may be drawn three perpendicular axes of symmetry of S_2. It then follows from the definition of S_2 and elementary symmetry considerations that $\mathbf{T}_2^0$ is a diagonal tensor and that

$$\mathrm{grad}\,\mathbf{T}_2 = \mathrm{grad}^0\,\mathbf{T}_2 = 0$$

at the point O. Therefore in this case equations (11) and (13) are the more accurate ones. This type of situation occurs in the motion of a body down the axis of symmetry of a circular cylinder or mid-way between two parallel plates. In these cases, the accuracy of the form of the resultant force is independent of the properties of symmetry of S_1.

We consider now the case when the boundary is completely asymmetric and we shall assume that S_1 possesses three perpendicular planes of symmetry intersecting at a point O within itself. The normals to the planes of symmetry will be taken to be the co-ordinate axes Ox_1, Ox_2 and Ox_3 and it will also be assumed that S_1 is moving parallel to Ox_1. The affix zero will be employed to denote the co-ordinates of the arbitrary P_0 on S_1. If the Stokes problem for S_1 in unbounded fluid is formulated as an integral equation, it follows from the definition of $\mathbf{T}_1$ and the symmetry properties of S_1 that the following relationships are valid:

$$\left.\begin{aligned} f_1^\infty(x_1^0,x_2^0,x_3^0) &= f_1^\infty(-x_1^0,x_2^0,x_3^0) = f_1^\infty(x_1^0,-x_2^0,x_3^0),\\ f_2^\infty(x_1^0,x_2^0,x_3^0) &= -f_2^\infty(-x_1^0,x_2^0,x_3^0) = -f_2^\infty(x_1^0,-x_2^0,x_3^0) = f_2^\infty(x_1^0,x_2^0,-x_3^0),\\ f_3^\infty(x_1^0,x_2^0,x_3^0) &= -f_3^\infty(x_1^0,x_2^0,x_3^0) = f_3^\infty(x_1^0,-x_2^0,x_3^0). \end{aligned}\right\} \quad (16)$$

It follows immediately from equations (16) that

$$\int_{S_1} \mathbf{f}^\infty . \mathbf{r}_0 \, dS = 0$$

and hence that the second terms of equation (15) are identically zero. The first set of terms in equation (15), however, are non-zero and thus, neglecting terms of $O(\epsilon^3)$, $\mathbf{f}$ will be a superposition of the solution of equation (9) and the solutions $\mathbf{f}^*$ of equations of the form

$$x_r \mathbf{i}_s = \int_{S_1} \mathbf{f}^* . \mathbf{T}_1 \, dS, \quad (17)$$

where $\mathbf{i}_s$ is the unit vector parallel to Ox_s. It is sufficient to examine the case $s = 1$ and $r = 1$ or 2; the other cases may be treated similarly. Considering first $r = 1$, it follows from equation (17) and the symmetry conditions that

$$\left.\begin{aligned} f_1^*(x_1^0,x_2^0,x_3^0) &= -f_1^*(-x_1^0,x_2^0,x_3^0) = f_1^*(x_1^0,-x_2^0,x_3^0),\\ f_2^*(x_1^0,x_2^0,x_3^0) &= f_2^*(-x_1^0,x_2^0,x_3^0) = -f_2^*(x_1^0,-x_2^0,x_3^0),\\ f_3^*(x_1^0,x_2^0,x_3^0) &= f_3^*(x_1^0,-x_2^0,x_3^0) = -f_3^*(x_1^0,x_2^0,-x_3^0). \end{aligned}\right\} \quad (18)$$

For $r = 2$ it can be shown that

$$\left.\begin{aligned} f_1^*(x_1^0,x_2^0,x_3^0) &= f_1^*(-x_1^0,x_2^0,x_3^0) = -f_1^*(x_1^0,-x_2^0,x_3^0),\\ f_2^*(x_1^0,x_2^0,x_3^0) &= -f_2^*(-x_1^0,x_2^0,x_3^0) = f_2^*(x_1^0,-x_2^0,x_3^0),\\ f_3^*(x_1^0,x_2^0,x_3^0) &= f_3^*(-x_1^0,x_2^0,x_3^0) = -f_3^*(x_1^0,-x_2^0,x_3^0). \end{aligned}\right\} \quad (19)$$

It follows from equations (18) and (19) that all solutions of equation (17) satisfy the condition $\int_{S_1} \mathbf{f}^* \, dS = 0$ and hence that the error in equation (10) is $O(\epsilon^3)$. Hence for the two particular cases considered here equations (11) and (13) are correct to $O(\epsilon^3)$. The integral-equation approach thus gives both a simple method for deriving equation (10) and a method for verifying its accuracy.

3. The motion of two bodies

The integral-equation approach lends itself very readily to the formulation of Stokes flow problems in the region exterior to two (or more) bodies. This type of problem has been examined by several authors (Smoluchowski 1911, Burgers 1942, Kynch 1959, Brenner 1964). The bodies will be denoted by S_3 and S_4 and the appropriate generalization of equation (5) is

$$\mathbf{q} = \frac{1}{8\pi\mu}\int_{S_3} \mathbf{f}_3 . \mathbf{T}_1 dS - \frac{1}{8\pi\mu}\int_{S_4} \mathbf{f}_4 . \mathbf{T}_1 dS, \tag{20}$$

where $\mathbf{f}_3$ and $\mathbf{f}_4$ are the values of $\mathbf{f}$ on S_3 and S_4. It will be assumed that both S_3 and S_4 have the same characteristic dimension c, and that l denotes the distance between some point O of S_3 and some point O' of S_4. The velocities of S_3 and S_4 will be denoted by $\mathbf{q}_3$ and $\mathbf{q}_4$ and the unit vector parallel to OO' will be denoted by $\mathbf{i}$.

On neglecting terms of $O(c^2/l^2)$ it follows that the problem reduces to that of obtaining the solution of the equations

$$\left.\begin{aligned} \mathbf{q}_3 &= \frac{1}{8\pi\mu}\int_{S_3} \mathbf{f}_3 . \mathbf{T}_1 - \frac{1}{8\pi\mu l}[\mathbf{F}_4 + (\mathbf{F}_4 . \mathbf{i})\,\mathbf{i}], \quad \text{on} \quad S_3 \\ \mathbf{q}_4 &= \frac{1}{8\pi\mu}\int_{S_4} \mathbf{f}_4 . \mathbf{T}_1 - \frac{1}{8\pi\mu l}[\mathbf{F}_3 + (\mathbf{F}_3 . \mathbf{i})\mathbf{i}], \quad \text{on} \quad S_4, \end{aligned}\right\} \tag{21}$$

where $\mathbf{F}_3$ and $\mathbf{F}_4$ are the forces on S_3 and S_4, respectively. Thus, if $\mathbf{q}_3$ and $\mathbf{q}_4$ are given and the Stokes resistance dyadics $\mathbf{\Phi}_3$ and $\mathbf{\Phi}_4$ of S_2 and S_4, respectively, are known, two equations may be obtained for $\mathbf{F}_3$ and $\mathbf{F}_4$. These latter equations may be solved and a result obtained by Brenner (1964) will then be re-derived. In problems involving settling particles, however, the forces $\mathbf{F}_3$ and $\mathbf{F}_4$ are known and the problem is to determine $\mathbf{q}_3$ and $\mathbf{q}_4$. If $\mathbf{u}_3$ denotes the settling velocity in a given force field of S_3 in an unbounded medium, then $\mathbf{F}_3$ in equation (21) will be defined by $\mathbf{F}_3 = -6\pi\mu c \mathbf{u}_3 . \mathbf{\Phi}_3$, and a similar relation holds between $\mathbf{F}_4$, $\mathbf{\Phi}_4$ and $\mathbf{u}_4$. Hence from equation (21)

$$\mathbf{q}_3 - \frac{3}{4}\frac{c}{l}[\mathbf{u}_4 . \mathbf{\Phi}_4 + (\mathbf{u}_4 . \mathbf{\Phi}_4) . \mathbf{ii}] = -\frac{1}{8\pi\mu}\int_{S_3} \mathbf{f}_3 . \mathbf{T}_1, \tag{22}$$

$\mathbf{f}_3$ is, however, the solution of equation (22) with the left-hand side equal to the settling velocity appropriate to the force $\mathbf{F}_3$ (i.e. $\mathbf{u}_3$). Thus

$$\mathbf{q}_3 = \mathbf{u}_3 + \frac{3}{4}\frac{c}{l}[\mathbf{u}_4 . \mathbf{\Phi}_4 + (\mathbf{u}_4 . \mathbf{\Phi}_4) . \mathbf{ii}]. \tag{23}$$

Equation (23) is a generalization of a result obtained by Kynch and Brenner by somewhat more elaborate calculation.

REFERENCES

BRENNER, H. 1962 *J. Fluid Mech.* **12**, 35.

BRENNER, H. 1963 *J. Chem. Engng Sci.* **18**, 1.

BRENNER, H. 1964 *J. Fluid Mech.* **18**, 144.

BRENNER, H. & HAPPEL, J. 1958 *J. Fluid Mech.* **4**, 195.

BURGERS, J. M. 1942 *Proc. K. Ned. Akad. Wet.* **43**, 307, 425, 625.

KYNCH, G. J. 1959 *J. Fluid Mech.* **5**, 193.

OSEEN, C. W. 1927 *Neuere Methoden und Ergebnisse in der Hydrodynamik*, §3. Leipzig Akademische Verlagsgesellschaft.

SMOLUCHOWSKI, M. A. 1911 *Bull. Acad. Sci. Cracow*, **1** *a*, 28.

WILLIAMS, W. E. 1964 *Proc. Edinb. Math. Soc.* **14**, 89.